Study of the GBM Auction

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1 Abstract

First published in 2018 and introduced on the NFT marketplace Cryptograph in 2020 [1], the Gonnaud–Bessire–McDonaugh ("GBM") auction is an open ascending auction model in which bidders are paid an incentive when they are outbid. In short, a GBM auction participant either wins the auction or leaves with more money than they started with.

In this paper, we first mathematically define the GBM auction and find the bidding patterns which maximise and minimise the revenue for the seller as a percentage of the winning bid. We express those best and worse revenue cases as a function of the GBM auction parameters, and verify our findings through a numerical approach.

We then demonstrate that the GBM auction is a generalisation of the English auction, and study how some of the key concepts and theorems in auction theory literature apply to the GBM auction.

Finally, we explore some of the properties and variations of the GBM auction, including other incentive functions and a multi-unit variation of the GBM auction.

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2 The GBM Auction

A Gonnaud–Bessire–McDonaugh (”GBM”) auction is an open ascending auction in which bidders are paid an incentive when they are outbid.

From a starting bid[1][2], each new bid needs to be higher than the previous bid by a minimum step. All bids are public. When a bidder is outbid, they are paid an incentive in addition to being refunded the full amount of their bid. The auction is time-limited, and the highest bidder at the end of the auction wins. Additional time can be added when a new bid is placed towards the end of the auction.

The incentive paid to a bidder when they are outbid can be calculated as a percentage of their bid, and can also depend on the value of the previous standing bid or the ratio between the previous bid and their bid. Because of this incentive model, bids must be fully funded (i.e. bidders commit the full amount of their bid when they place a bid) in order to ensure the solvability of the auction.

In this paper, we study the following incentive function g, which takes as parameters the previous standing bid and the new bid, and returns the incentive that a new bidder will receive when they are outbid, as a percentage of the value of that new bid.

$g : (x, y) \longrightarrow min \left[inc_{max}, inc_{min} + a \times \frac{y - x(1+step_{min})}{x(1+step_{min})} \right]$ for $x > 0$ and
$g(0, y) = inc_{max}$

with :

- x : the previous standing bid.
- y : the new bid which is going to replace the standing bid.
- inc_{max} : the maximum incentive, expressed as a percentage of the new bid amount.
- inc_{min} : the minimum incentive, expressed as a percentage of the new bid amount.
- a : the accelerator, expressed as a positive real number. It controls the growth of the percentage of the new bid that is paid as incentive, relative to the ratio between the previous standing bid and the new bid.
- $step_{min}$: the minimum bid increment, expressed as a percentage of the previous standing bid.

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[1] The starting bid is the lowest amount the first bidder can bid. The starting bid is defined by the seller and can be 0.

[2] Because of the incentive model of the GBM auction, the first bid may be different than the starting bid.

NB: The incentive that the new bidder will receive when (if) they are outbid is calculated the moment the new bidder places the new bid. This is because the incentive function g here only depends on the previous standing bid and the new bid.

Example:

1. Bidder 1 places the first bid b_1. Bidder 1 being the first bidder, the previous bid is equal to 0. The standing bid is now b_1, there is b_1 in the pot, and the incentive that Bidder 1 will receive when (if) outbid by Bidder 2 is: $b_1 \times \mathrm{g}(0,\mathrm{b}_1) = b_1 \times \ inc_{max}$
2. Bidder 2 then places a bid b_2. Bidder 1 receives $b_1 \times \ inc_{max}$ in incentive. The standing bid is now b_2, there is $(b_2 - b_1 \times \ inc_{max})$ in the pot, and the incentive that Bidder 2 will receive when (if) outbid by Bidder 3 is: $b_2 \times \ g(b_1, b_2)$
3. Bidder 3 then places a bid b_3. Bidder 2 receives $b_2 \times \ g(b_1, b_2)$ in incentive. The standing bid is now b_2, there is $(b_3 - b_2 \times \ g(b_1, b_2) - b_1 \times \ inc_{max})$ in the pot, and the incentive that Bidder 3 will receive when (if) outbid by Bidder 3 is: $b_3 \times \ g(b_2, b_3)$
4. There are no further bids. The auction ends with b_3 as the winning bid. No incentives are paid to Bidder 3. The seller receives $(b_3 - b_2 \times \ g(b_1, b_2) - b_1 \times \ inc_{max})$.

3 GBM Auction Revenue

3.1 Objectives and Definitions

The revenue of an auction R is defined as the amount the seller receives from the auction. In a GBM auction, because money is redistributed to all but the last bidder, the parameters of the incentive function and the bidding pattern to reach the final price all influence the revenue.

In order to agree to sell an item using the GBM auction, a seller needs to understand how much of the last bid they would be receiving. Therefore, we wish to express the revenue of a GBM auction a percentage of the winning bid b_n, that we call $R_{\%}$. Although $R_{\%}$ will depend heavily on the bidding pattern and is as such unpredictable, what we may be able to provide the seller with the minimum and maximum values of $R_{\%}$, hence giving a seller an interval that they can use to make a decision on using a GBM auction.

Our objective is to find the minimum value and the maximum value of $R_{\%}$ as a function of inc_{min}, inc_{max}, $step_{min}$, a, the number of bids n, the first bid

b_1, and the last bid b_n.

A GBM auction is defined as a set of bids $b = \{b_1, \dots, b_n\}$, and the incentive function g as defined above in Section 2.

We introduce the constant $\alpha = 1 + step_{min}$, and as per the constraints of a GBM auction we have $b_{i+1} \geq \alpha b_i > 0$ for $i \geq 1$.

$g_i = g(b_{i-1}, b_i),\ i \in [\![2; n]\!]$ is the incentive percentage distributed to the i^{th} bidder when outbid. $g_1 = g(0, b_1) = inc_{max}$, and g_n is defined but irrelevant as incentives are never paid to the last bidder.

The revenue of the auction is equal to the winning bid minus the sum of the incentives distributed to bidders:

$$\begin{aligned} R &= R(b, inc_{max}, inc_{min}, a, step_{min}) \\ &= b_n - \sum_{i=1}^{n-1} g_i b_i \end{aligned}$$

The revenue of that auction as a percentage of the winning bid $R_{\%}$ is:

$$\begin{aligned} R_{\%} &= R_{\%}(b, inc_{max}, inc_{min}, a, step_{min}) \\ &= \frac{R}{b_n} \\ &= \frac{b_n - \sum_{i=1}^{n-1} g_i b_i}{b_n} \\ &= 1 - \frac{\sum_{i=1}^{n-1} g_i b_i}{b_n} \end{aligned}$$

To find the minimum value and the maximum value of $R_{\%}$, we need to find the bidding patterns $b_2 ... b_{n-1}$ that minimize (worst case) and maximize (best case) $R_{\%}$.

3.2 Equivalence to work with b_1, b_n or $\gamma b_1, \gamma b_n, \gamma \in \mathbb{R}_+^*$

Looking for $R_{\%}$, we do not really care about the values of b_1 and b_n, but rather about the ratio $\frac{b_n}{b_1}$. Let's demonstrate this.

First, we demonstrate that for a set of bids $b = \{b_1, \dots, b_n\}$, there is revenue equivalence in a GBM auction for b and $\tilde{b} = \{\gamma b_1, \dots, \gamma b_n\}$ with $\gamma \in \mathbb{R}_+^*$.

$$\begin{aligned}\widetilde{g_i} = g(\widetilde{b_{i-1}}, \widetilde{b_i}) &= min(inc_{max}, inc_{min} + a\left(\frac{\widetilde{b_i}}{\alpha\widetilde{b_{i-1}}} - 1\right)) \\ &= min(inc_{max}, inc_{min} + a\left(\frac{\gamma b_i}{\alpha\gamma b_{i-1}} - 1\right)) \\ &= min(inc_{max}, inc_{min} + a\left(\frac{b_i}{\alpha b_{i-1}} - 1\right)) \\ &= g_i(b_{i-1}, b_i) \\ &= g_i\end{aligned}$$

Thus:

$$\begin{aligned}R_{\%}(\widetilde{b}) &= 1 - \frac{\sum_{i=1}^{n-1}\widetilde{g_i}\widetilde{b_i}}{\widetilde{b_n}} \\ &= 1 - \frac{\gamma\left(\sum_{i=1}^{n-1} g_i b_i\right)}{\gamma b_n} \\ &= 1 - \frac{\sum_{i=1}^{n-1} g_i b_i}{b_n} \\ &= R_{\%}(b)\end{aligned}$$

Therefore, we have $R_{\%}(\widetilde{b}) = R_{\%}(b), \gamma \in \mathbb{R}_+^*$ (Q.E.D).

Because of this revenue equivalence, we now demonstrate that the worse and best bidding pattern for a given b_1, b_n and n will be equivalent to the worse and best bidding pattern for $\widetilde{b_1}$, $\widetilde{b_n}$ and n.

We denote by $\widetilde{b_i}$ the bids between $\widetilde{b_1}$ and $\widetilde{b_n}$ such that $\widetilde{b_i} \geq \alpha\widetilde{b_{i-1}}$, $\widetilde{b_2} \geq \alpha\widetilde{b_1} = \alpha\gamma b_1$ and $\widetilde{b_{n-1}} \leq \frac{\widetilde{b_n}}{\alpha} = \frac{\gamma b_n}{\alpha}$.

Then $\alpha\widetilde{b_1} = \gamma\alpha b_1 \leq \widetilde{b_2} \leq \frac{\widetilde{b_3}}{\alpha} \leq \ldots \leq \frac{\widetilde{b_n}}{\alpha^{n-2}} = \frac{\gamma b_n}{\alpha^{n-2}}$, and thus:

$$\begin{aligned}\gamma\alpha b_1 \leq \widetilde{b_2} \leq \frac{\gamma b_n}{\alpha^{n-2}} &\iff \alpha b_1 \leq \frac{\widetilde{b_2}}{\gamma} \leq \frac{b_n}{\alpha^{n-2}} \\ &\iff \alpha b_1 \leq b_2 \leq \frac{b_n}{\alpha^{n-2}}\end{aligned}$$

with $b_2 = \frac{\widetilde{b_2}}{\gamma}$. A similar reasoning gives $\alpha b_1 \leq \frac{\widetilde{b_i}}{\gamma} \leq \frac{b_n}{\alpha^{n-2}}$ for every i in $[\![3, n-1]\!]$. Then we can set $b_i = \frac{\widetilde{b_i}}{\gamma}$ and we get the inequalities only with the bids b_is. Therefore it is equivalent to work with the b_is (Q.E.D).

3.3 Trivial Cases

Here we treat the cases $n = 1$ and $n = 2$. In the following section we will focus on the cases where $n \geq 3$.

If n = 1:
There is only one bid, then $b_1 = b_n$ and $R_{\%min} = 1$ because no incentives are distributed (the first bid is the winning bid). The same applies for $R_{\%max}$: $R_{\%max} = 1$.

If n = 2:
The two bids are the bids b_1 and b_n, with $b_n \geq \alpha b_1$. Thus everything is fixed which implies $R_{\%max}$ and $R_{\%min}$ have the same value.

$$R_{\%max} = R_{\%min} = 1 - \frac{g(b_1, b_n)}{b_n} = 1 - \frac{inc_{max} b_1}{b_n}$$

NB: For all variables fixed except b_1, then the worse bidding pattern would be when $b_1 = \frac{b_n}{\alpha}$, for which the first bidder will receive the maximum inc_{max} percentage return on their bid. In this case, we would find $R_{\%min} = 1 - \frac{inc_{max}}{\alpha}$

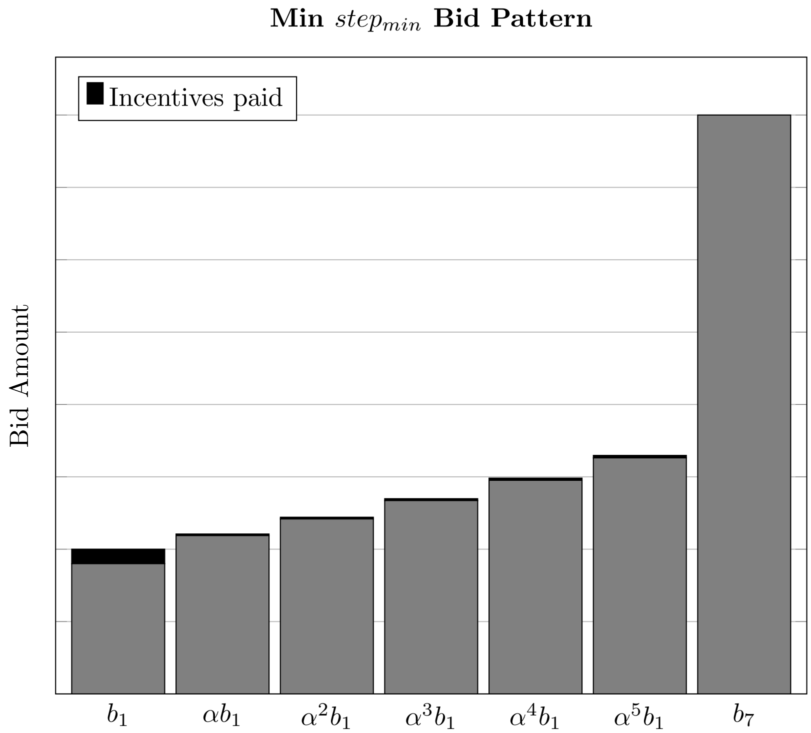
3.4 Bidding Patterns

Let us study bidding patterns that we think may lead to the worse or best case for the revenue, with b_1, b_n and n fixed.

3.4.1 Min $step_{min}$ Bid Pattern

From b_1, each new bid is of the minimum amount to follow the constraint of the incentive function g, except the last bid b_n.

That is, $b_{i+1} = \alpha b_i$ where (reminder: $\alpha = (1 + step_{min})$).



This pattern minimizes incentives paid to bidders. We show that it is the best pattern to maximize the revenue in percentage.

To maximize $R_{\%}$, it is equivalent to minimize the term $\frac{\sum_{i=1}^{n-1} g_i b_i}{b_n}$. With b_n set, we only need to minimize the numerator $\sum_{i=1}^{n-1} g_i b_i$. We can minimize every $g_i b_i$ by taking the minimum value for each b_i, that is to say $b_i = \alpha b_{i-1}$.

With these values we get $g_i = min(inc_{max}, inc_{min} + 0) = inc_{min}$ (which is the minimum value of every g_i). Then $\frac{\sum_{i=1}^{n-1} g_i b_i}{b_n}$ is minimized.

Thus the Min $step_{min}$ Bid Pattern is the best pattern to maximize $R_{\%}$. Now we express the maximum percentage that we can get from the revenue. We denote it by $R_{\%max}$.

We have $b_i = \alpha b_{i-1} = \alpha^2 b_{i-2} = \ldots = \alpha^{i-1} b_1$

Then, we can replace in $R_{\%max}$:

$$\begin{aligned}
R_{\%max} &= 1 - \frac{inc_{max}b_1 + \sum_{i=2}^{n-1} g_i b_i}{b_n} \\
&= 1 - \frac{inc_{max}b_1 + \sum_{i=2}^{n-1} inc_{min}\alpha^{i-1}b_1}{b_n} \\
&= 1 - \frac{inc_{max}b_1 + inc_{min}b_1((\sum_{i=0}^{n-2} \alpha^i) - 1)}{b_n} \\
&= 1 - \frac{inc_{max}b_1 + inc_{min}b_1(\frac{1-\alpha^{n-2}}{1-\alpha} - 1)}{b_n} \\
&= 1 - \frac{inc_{max}b_1 + inc_{min}b_1\alpha\frac{1-\alpha^{n-3}}{1-\alpha}}{b_n} \\
&= 1 - \frac{b_1}{b_n}(inc_{max} + inc_{min}\alpha\frac{\alpha^{n-3}-1}{\alpha-1})
\end{aligned}$$

Because $\forall i \in [\![1; n-1]\!]$, $b_{i+1} \geq \alpha b_i$, then there is a maximum number of bids the GBM auction can have for a given b_1 and b_n.

We introduce n_{max} as the maximal number of bids that a GBM auction using the incentive function g, with a first bid b_1 and a last bid b_n, can have. This n_{max} excludes the known bids b_1 and b_n, and only counts the bids between b_1 and b_n.

Therefore we are looking for $n_{max} \in \mathbb{N}$ such that $\alpha^{n_{max}}b_1 \leq b_n \leq \alpha^{n_{max}+1}b_1$.

$$\begin{aligned}
\alpha^{n_{max}}b_1 \leq b_n \leq \alpha^{n_{max}+1}b_1 &\iff \alpha^{n_{max}} \leq \frac{b_n}{b_1} \leq \alpha^{n_{max}+1} \\
&\iff n_{max}log(\alpha) \leq log\left(\frac{b_n}{b_1}\right) \leq (n_{max}+1)log(\alpha) \\
&\iff n_{max} \leq \frac{log(b_n) - log(b_1)}{log(\alpha)} \leq (n_{max}+1) \\
&\text{Thus, by definition of the floor function:} \\
&\iff n_{max} = \left\lfloor \frac{log(b_n) - log(b_1)}{log(\alpha)} \right\rfloor
\end{aligned}$$

For example if $\alpha = 1.105$, $b_1 = 1$ and $b_n = 10$ the GBM auction can have at most $\left\lfloor \frac{log(10)-log(1)}{log(1.105)} \right\rfloor = 23$ bids between b_1 and b_n. We can check it with computation:

$$\alpha^{23}b_1 \simeq 9.939 \leq b_n \text{ and } \alpha^{24}b_1 \simeq 10.98 > b_n$$

Now let us pick a n $\in \mathbb{N}, n \geq 3$ and let us distinguish between cases.

If $n > n_{max}$: then there exists no bidding pattern following the constraints i.e. the auction is not possible.

If $n = n_{max}$:
With the constraints we are able to bound every bid b_i : $b_i \in [\alpha b_{i-1}, \frac{b_{i+1}}{\alpha}]$ for i $= 2, \dots, n-2$ and $b_{n-1} \in [\alpha b_{n-2}, \frac{b_n}{\alpha}]$

Let us suppose that $\alpha^{n-1} b_1 = b_n$, that is to say b_n is **reached exactly**.

We demonstrate that the current pattern is the best pattern by contradiction. Let us suppose that there exists another pattern that the Min $step_{min}$ Bid pattern with e = 0 which minimizes our revenue. Then there exists a set J $\subset [\![2, n-1]\!]$ $(J \neq \varnothing)$ such that for every j $\in J$, $b_j > \alpha b_{j-1}$, and let β_J the minimum value of these b_js such that $b_j = \beta_J b_{j-1}$ (of course $\beta_J > \alpha$). Then $b_{n-1} \geq \beta_J^{|J|} \alpha^{n-|J|-1} b_1 > \alpha^{|J|} \alpha^{n-|J|-1} b_1 = \alpha^{n-1} b_1 = b_n$ thus we come to a contradiction. Thus the Min $step_{min}$ Bid pattern is the best that we can achieve to get the minimum percentage for the revenue. Thus $R_{\%min}$ is the same than $R_{\%max}$, i.e:

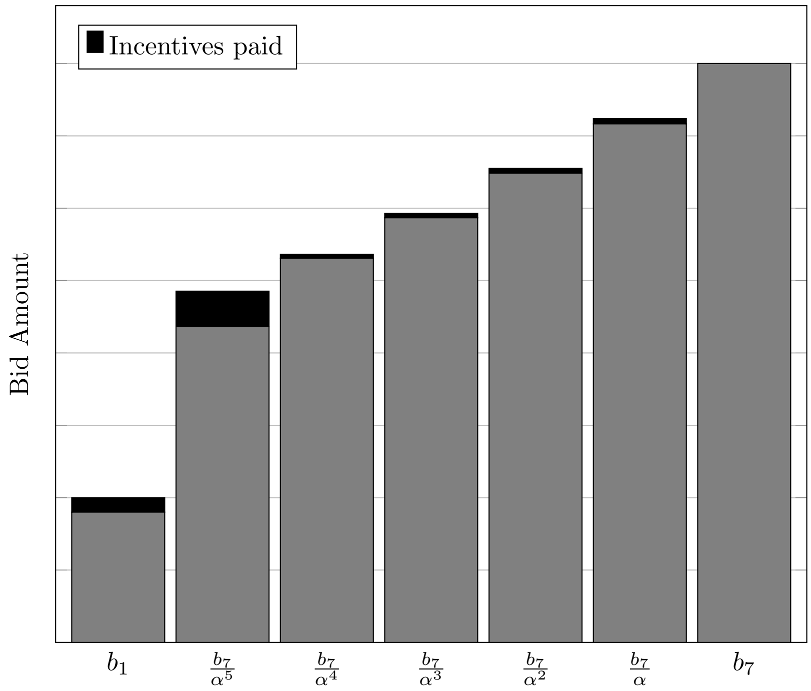
$$R_{\%min} = 1 - \frac{b_1}{b_n}(inc_{max} + inc_{min}\alpha \frac{\alpha^{n-3} - 1}{\alpha - 1})$$

Now we suppose that b_n is **not reached**.

We can introduce $e \in \mathbb{R}_+^*$ such that $\alpha b_{n-1} + e = b_n \iff \alpha^{n-1} b_1 + e = b_n$. As $\alpha b_{n-1} \leq b_n$, we know that $e \geq \alpha b_{n-1} - \alpha^{n-1} b_1$. But we need to be careful with this pattern, because it presents one limitation.
We want $b_n = e + b_{n-1}$. One possibility is to cut out e in n parts such that $b_i = \alpha b_{i-1} + e_n$. In this case we find that $e = e_n \left(\frac{1-\alpha^{n-1}}{1-\alpha}\right) \iff e_n = \frac{e(\alpha-1)}{\alpha^{n-1}-1}$.
Then we have just found another pattern which may give a greater percentage than the percentage that we get with the Min $step_{min}$ Bid pattern.

3.4.2 Max $step_{min}$ Bid Pattern

From b_2, every bid is so that every new bid is the minimum bid and b_n is reached, that is $(b_i = \frac{b_{i+1}}{\alpha})$ until b_2.
We can see this pattern as the Min $step_{min}$ Bid Pattern, but the other way around, because we do not start from b_1 but from b_n and we divide by α instead of multiply by the latter.


Max $step_{min}$ Bid Pattern

At first sight we have the choice between the Min $step_{min}$ Bid and the Max $step_{min}$ Bid pattern. The first one takes the minimum values for each b_i, that is to say αb_{i-1} while the second one uses the maximum values for each b_i, i.e $b_i = \frac{b_{i+1}}{\alpha}$. As we want to maximize the sum $\sum_{i=1}^{n-1} g_i b_i$ we have an interest to maximize the b_is. Thus the second model, the Max $step_{min}$ Bid pattern is better than the first one.

But we need to be careful with this pattern, because it presents a restrictive limitation.
This limit is about g_i (reminder $g_i = g(b_{i-1}, b_i)$). Within the Max $step_{min}$ pattern of course we get the maximum for the b_is, but it is not the case of the g_is. By taking $b_i = \frac{b_{i+1}}{\alpha}$ we maximize b_i but we minimize g_i :

$$g_i = min\left[inc_{max}, inc_{min} + a\left(\frac{\alpha b_i}{\alpha b_{i-1}} - 1\right)\right] = inc_{min}$$

With this limitation the Max $step_{min}$ Bid pattern may not be the best pattern to get the highest percentage. However we can formulate the result for $R_{\%min}$ when choosing the Max $step_{min}$ Bid pattern, in the case where this pattern is the best candidate that we can achieve in theory.

In order to do that we need to write b_i based on b_n, α, n and i:

$$b_{n-1} = \frac{b_n}{\alpha}, b_{n-2} = \frac{b_{n-1}}{\alpha} = \frac{b_n}{\alpha^2}, b_{n-3} = \frac{b_n}{\alpha^3}, \dots, b_2 = \frac{b_n}{\alpha^{n-2}}$$

By immediate recursion we deduce that $b_i = \frac{b_n}{\alpha^{n-i}}$.
Now we rewrite the percentage $R_{\%min}$ when choosing the Max $step_{min}$ Bid pattern:

$$\begin{aligned}
R_{\%min} &= 1 - \frac{inc_{max} b_1 + \sum_{i=2}^{n-1} g_i b_i}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + \sum_{i=2}^{n-1} inc_{min} b_i}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{min} \sum_{i=2}^{n-1} b_i}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{min} \sum_{i=2}^{n-1} \frac{b_n}{\alpha^{n-i}}}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{min} \frac{b_n}{\alpha^n} \sum_{i=2}^{n-1} \alpha^i}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{min} \frac{b_n}{\alpha^n} \left(\sum_{i=0}^{n-1} \alpha^i - 1 - \alpha \right)}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{min} \frac{b_n}{\alpha^n} \left(\frac{1-\alpha^{n-1}}{1-\alpha} - 1 - \alpha \right)}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{min} \frac{b_n}{\alpha^{n-1}} \left(\frac{1-\alpha^{n-2}}{1-\alpha} - 1 \right)}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{min} \frac{b_n}{\alpha^{n-2}} \frac{1-\alpha^{n-3}}{1-\alpha}}{b_n} \\
&= 1 - \frac{inc_{max} b_1}{b_n} - \frac{inc_{min} \frac{b_n}{\alpha^{n-2}} \frac{\alpha^{n-3}-1}{\alpha-1}}{b_n} \\
&= 1 - \frac{inc_{max} b_1}{b_n} - inc_{min} \frac{\alpha^{n-3} - 1}{\alpha^{n-2}(\alpha - 1)}
\end{aligned}$$

Now we need to go deeper to find (if possible) a best pattern.
Always with $n = n_{max}$, we are looking for the minimum value between the b_is which maximizes g_i (reminder the maximum of g is inc_{max}). We denote it Δ.

Let's find Δ:

$$\begin{aligned}
inc_{min} + a\frac{b_i - \alpha b_{i-1}}{b_{i-1}\alpha} \geq inc_{max} &\iff a\left(\frac{b_i}{\alpha b_{i-1}} - 1\right) \geq inc_{max} - inc_{min} \\
&\iff \frac{b_i}{\alpha b_{i-1}} \geq \frac{inc_{max} - inc_{min}}{a} + 1 \\
&\iff b_i \geq \left(\frac{inc_{max} - inc_{min}}{a} + 1\right)\alpha b_{i-1} \\
&\iff b_i \geq \Delta b_{i-1} \ with \ \Delta = \alpha\left(\frac{inc_{max} - inc_{min}}{a} + 1\right)
\end{aligned}$$

Let $D_i = \{b_i \in \mathbb{R}, b_i \geq \Delta b_{i-1}\}$ for i $= 2, \ldots, n-1$.
Our function $R_{\%}$ is continuous everywhere over $\mathbb{R}^{n-2}$ but is not differentiable everywhere, because of the function min. In fact for each derivative we need to distinguish between cases whether $b_i \geq \Delta b_{i-1}$ for the general case. As n $=$ n_{max}, we know there is no $i \in [\![2, n-1]\!]$ such that $b_i \geq \Delta b_{i-1}$.
In other words for all i $\in [\![2, n-2]\!]$, $b_i \in \overline{D_i}$. In this case we have partial derivatives which are continuous and equal to :

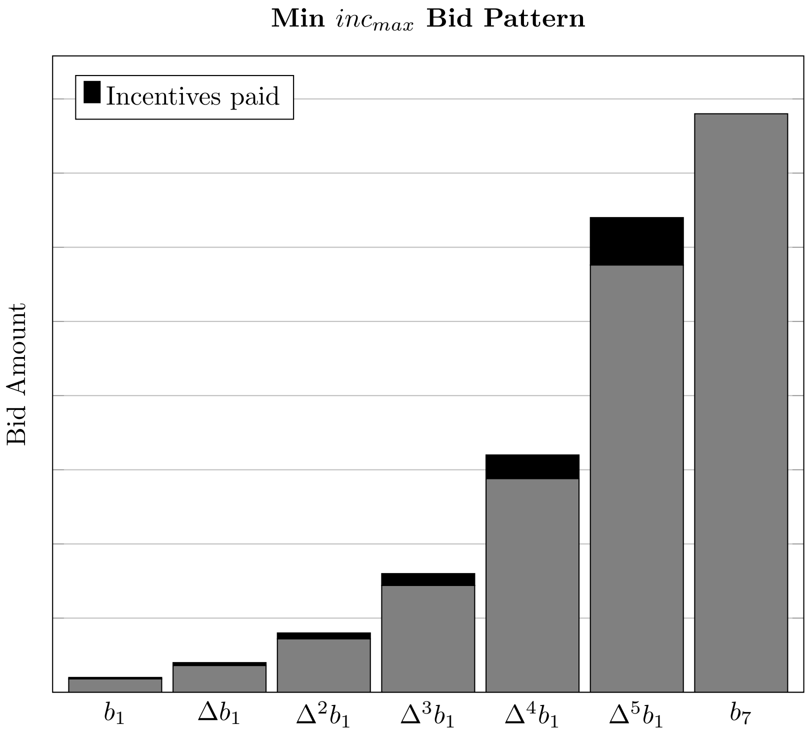
$$\forall i \in [\![2, n-2]\!],\ 2\frac{b_i}{b_{i-1}} - \frac{b_{i+1}^2}{b_i^2} = \frac{\alpha}{a}(a - inc_{min}) \qquad (*)$$

We can start from b_{n-1} for the formula and take the greatest value, that is to say $b_{n-1} = \frac{b_n}{\alpha}$. Then we solve the equation (*).
This equation has no explicit result, this is why we decided to solve it numerically.

For $n < n_{max}$ we need to find and to study new patterns, to figure out $R_{\%min}$.

3.4.3 Min inc_{max} Bid Pattern

In this pattern, we start from b_1, and each new bid is of the minimum amount required for the bidder to maximize the percentage return on their bid (i.e. $g(b_{i-1}, b_i) = inc_{max}$, except the last bid b_n.
In fact we have $b_{i+1} = \Delta b_i$ for i= $1 \ldots n-2$.


Min inc_{max} Bid Pattern

As $b_{i+1} = \Delta b_i$ for i$= 1 \dots n-2$, we find that $b_i = \Delta^{i-1} b_1, i = 2, \dots, n-1$. Again, we can introduce $e \in \mathbb{R}$ such that $b_n = \Delta^{n-2} b_1 + e$, with $e < b_n - \Delta^{n-2} b_1$.

If b_n is **reached exactly** (that is to say $b_n = \Delta^{n-1} b_1$), we have a perfect Min inc_{max} bid pattern with $e = 0$ (the proof is similar to the one written at the end of the first pattern). Otherwise some patterns may be better like the next one for $n < n_{max}$.

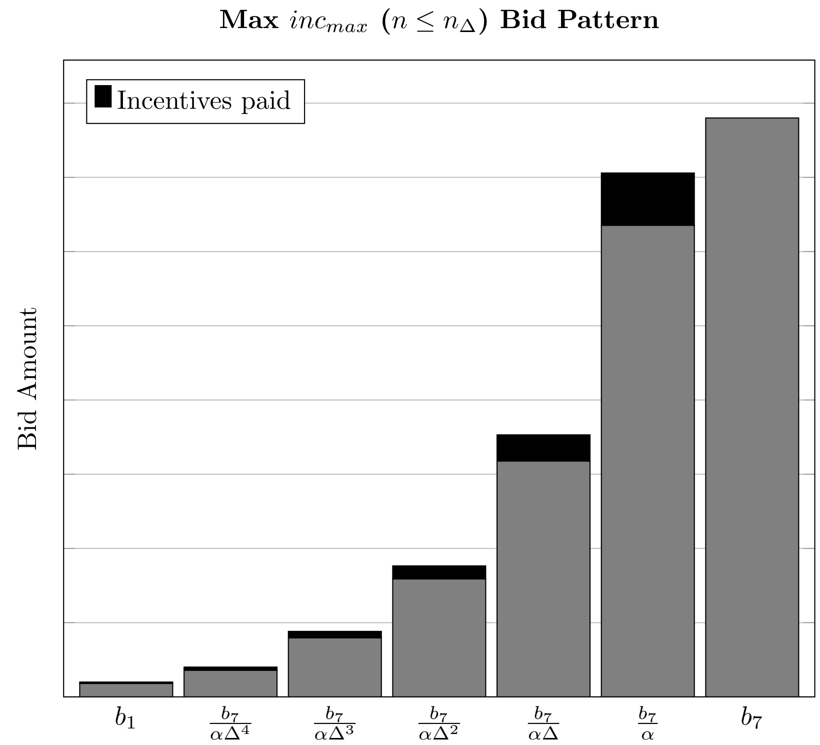
For $R_{\%min}$ we get:

$$\begin{aligned}
R_{\%min} &= 1 - \frac{inc_{max} b_1 + \sum_{i=2}^{n-1} g_i b_i}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{max} \sum_{i=2}^{n-1} b_i}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{max} \sum_{i=2}^{n-1} \Delta^{i-1} b_1}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{max} \left(\sum_{i=0}^{n-2} \Delta^{i} - 1\right) b_1}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{max} b_1 \left(\frac{1-\Delta^{n-1}}{1-\Delta} - 1\right)}{b_n} \\
&= 1 - inc_{max} b_1 \frac{1 + \frac{1-\Delta^{n-1}}{1-\Delta} - 1}{b_n} \\
&= 1 - inc_{max} \frac{(\Delta^{n-1} - 1) b_1}{(\Delta - 1) b_n}
\end{aligned}$$

3.4.4 Max inc_{max} Bid Pattern

From b_n, the penultimate bid b_{n-1} is of the maximum possible amount so that every subsequent bid before b_{n-1} is of the minimum amount required for each bidder to maximize the percentage return on their bid and the penultimate bid is such that only the minimum bid is required to reach b_n

That is to say, $b_{n-1} = \frac{b_n}{\alpha}$ and $b_{i-1} = \frac{b_i}{\Delta}$ for i= $2 \ldots n-2$.
We take $b_{n-1} = \frac{b_n}{\alpha}$ to maximize the values of the b_is and thus minimize the numerator in $R_{\%}$.
This pattern seems to be a good candidate to maximise the incentives paid to bidders and thus minimise $R_{\%}$.



$\forall i \in [\![2, n-2]\!]$, $b_i = \frac{b_{i+1}}{\Delta}$ and $b_{n-1} = \frac{b_n}{\alpha}$

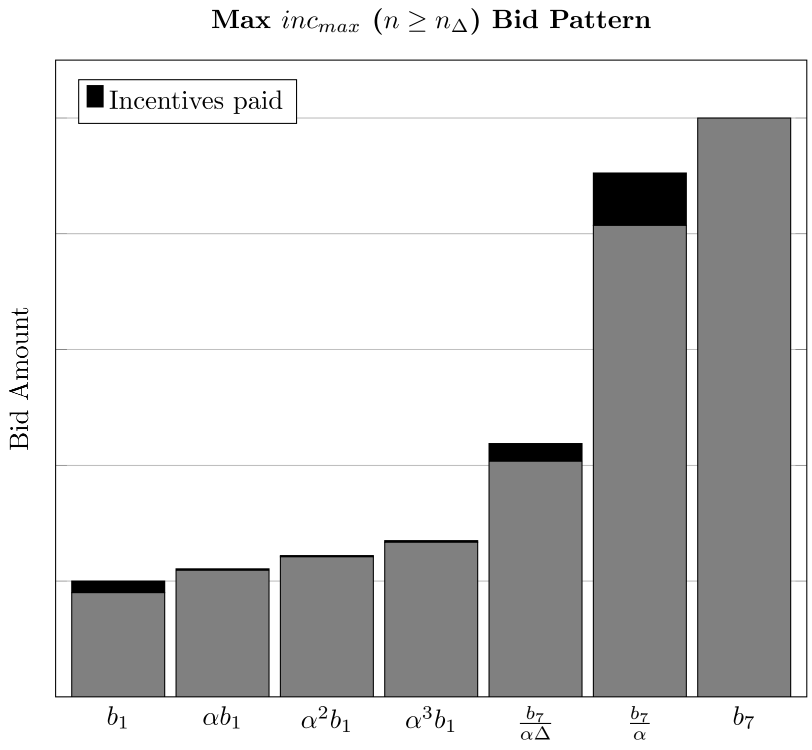
For a given n, b_1 and b_n we cannot divide every time by Δ and respect the constraints, because we have a similar problem than the one in the first pattern, it exists a sort of limit of bids that we can make between b_1 and b_n such that $b_i = \Delta b_{i-1}, i = 2, \ldots, n-2$. This is why we define n_Δ, the maximal number of bids that a GBM auction can have, with the bids which follow the following constraints : $b_i \geq \Delta b_{i-1} \forall i \in [\![2, n-2]\!]$.
In fact it represents the maximal number of bids that we can make which maximize g_i.
Let us find n_Δ.

$$\begin{aligned}
\Delta^{n_\Delta} b_1 \leq \frac{b_n}{\alpha} \leq \Delta^{\Delta+1} b_1 &\iff \Delta^{n_\Delta} \leq \frac{b_n}{\alpha b_1} \leq \Delta^{n_\Delta+1} \\
&\iff n_\Delta log(\Delta) \leq log\left(\frac{b_n}{\alpha b_1}\right) \leq (n_\Delta + 1) log(\Delta) \\
&\iff n_\Delta \leq \frac{log(b_n) - log(\alpha b_1)}{log(\Delta)} \leq (n_\Delta + 1)(\text{floor function}) \\
&\iff n_\Delta = \left\lfloor \frac{log(b_n) - log(\alpha b_1)}{log(\Delta)} \right\rfloor
\end{aligned}$$

For example for $b_1 = 1$ and $b_n = 100$, $inc_{min} = 0.01$, $inc_{max} = 0.1$, $a = 0.1112$, $\alpha = 1.105$ we find $n_\Delta = 6$, that is to say we can only make 6 bids respecting the condition $b_i \geq \Delta b_{i-1}$. Thus if $n \leq n_\Delta$ the Max inc_{max} Bid pattern seems to be a good candidate to minimize the percentage of the revenue. With this pattern we get for $R_{\%min}$:

$$\begin{aligned}
R_{\%min} &= 1 - \frac{inc_{max} b_1 + \sum_{i=2}^{n-1} g_i b_i}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + \sum_{i=2}^{n-1} inc_{max} b_i}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + inc_{max} \sum_{i=2}^{n-1} \frac{b_n}{\alpha \Delta^{n-1-i}}}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + \frac{inc_{max} b_n}{\alpha \Delta^{n}} \sum_{i=2}^{n-1} \Delta^{i+1}}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + \frac{inc_{max} b_n}{\alpha \Delta^{n-3}} \sum_{i=2}^{n-1} \Delta^{i-2}}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + \frac{inc_{max} b_n}{\alpha \Delta^{n-3}} \sum_{i=0}^{n-3} \Delta^{i}}{b_n} \\
&= 1 - \frac{inc_{max} b_1 + \frac{inc_{max} b_n}{\alpha \Delta^{n-3}} \frac{1-\Delta^{n-2}}{1-\Delta}}{b_n} \\
&= 1 - inc_{max} \left(\frac{b_1}{b_n} + \frac{\frac{\Delta^{n-2}-1}{\alpha \Delta^{n-3}(\Delta-1)} b_n}{b_n} \right) \\
&= 1 - inc_{max} \left(\frac{b_1}{b_n} + \frac{\Delta^{n-2}-1}{\alpha \Delta^{n-3}(\Delta-1)} \right)
\end{aligned}$$

The results above apply to when we have less than n_Δ bids. Let's now determine the bidding pattern when we have more than n_Δ bids. One solution is to make the last bids follow the condition $b_i = \Delta b_{i-1}$ to get high values and high percentages, and the first bids got the minimal values, that is to say $b_i = \alpha b_{i-1}$ (because we have too many bids and not all the bids can follow the first constraint). This is what the following pattern achieves.



As $n_\Delta < n < n_{max}$, we can try to find an integer j to respect at the minimum the constraints to maximize the g_is, that is to say find j $\in [\![0, n-2]\!]$ such that $b_i = \frac{b_{i+1}}{\Delta}, i = n-2-j, \ldots, n-2$, $b_i = \alpha b_{i-1}, i = 2, \ldots, n-2-j$.
With i = j we get $b_j \alpha(b_{j-1}) = \alpha(\alpha^2 b_{j-2}) = \ldots = \alpha(\alpha^{n-j-2} b_1)$ Also we have $b_j = \frac{b_{j+1}}{\Delta} = \frac{b_{j+2}}{\Delta^2} = \ldots = \frac{b_n}{\alpha\Delta^j}$.
Thus we are looking for j such that $\frac{b_n}{\alpha\Delta^j} = \alpha(\alpha^{n-j-2})b_1$:

$$\begin{aligned}
b_j = \alpha b_{j-1} &\iff \frac{b_n}{\alpha\Delta^j} = \alpha(\alpha^{n-j-2} b_1) \\
&\iff \frac{b_n}{\Delta^j} = \alpha^{n-j} b_1 \\
&\iff \left(\frac{\alpha}{\Delta}\right)^j = \frac{\alpha^n b_1}{b_n} \\
&\implies j = \left\lfloor \frac{log(\alpha^n b_1) - log(b_n)}{log(\alpha) - log(\Delta)} \right\rfloor
\end{aligned}$$

We rewrite $R_{\%min}$:

$$\begin{aligned}
R_{\%min} &= 1 - \frac{inc_{max}b_1 + \sum_{i=2}^{n-1} g_i b_i}{b_n} \\
&= 1 - \frac{inc_{max}b_1 + \sum_{i=2}^{n-j-2} inc_{min} b_i + \sum_{i=n-j-1}^{n-2} inc_{max} b_i}{b_n} \\
&= 1 - \frac{inc_{max}b_1 + \sum_{i=2}^{n-j-2} inc_{min} \alpha^{i-1} b_1 + \sum_{i=n-j-1}^{n-2} inc_{max} \frac{b_n}{\Delta^{n-i}}}{b_n} \\
&= 1 - \frac{inc_{max}b_1 + inc_{min} b_1 (\sum_{i=0}^{n-j-3} \alpha^i - 1) + \frac{inc_{max} b_n}{\Delta^n} \sum_{i=n-j-1}^{n-2} \Delta^i}{b_n} \\
&= 1 - \frac{inc_{max}b_1 + inc_{min} b_1 (\frac{1-\alpha^{n-j-2}}{1-\alpha} - 1) + \frac{inc_{max} b_n}{\Delta^n} \sum_{i=0}^{n-2-(n-j-1)} \Delta^{i+(n-j-1)}}{b_n} \\
&= 1 - \frac{inc_{max}b_1 + inc_{min} \alpha (\frac{\alpha^{n-j-3}-1}{\alpha-1}) b_1 + \frac{inc_{max} b_n}{\Delta^{j+1}} \sum_{i=0}^{j-1} \Delta^i}{b_n} \\
&= 1 - \frac{inc_{max}b_1}{b_n} - \frac{inc_{min} \alpha (\frac{\alpha^{n-j-3}-1}{\alpha-1}) b_1}{b_n} - \frac{\frac{inc_{max}}{\Delta^{j+1}} (\frac{\Delta^j - 1}{\Delta - 1}) b_n}{b_n} \\
&= 1 - \frac{b_1}{b_n} \left(inc_{max} + inc_{min} \alpha \left(\frac{\alpha^{n-j-3} - 1}{\alpha - 1} \right) \right) - \frac{inc_{max}}{\Delta^{j+1}} \left(\frac{\Delta^j - 1}{\Delta - 1} \right)
\end{aligned}$$

With this j we get the Max inc_{max} Bid Pattern for $n > n_\Delta$. In fact it is a generalization of the Max inc_{max} Bid Pattern ($n \leq n_\Delta$), now for $n \in [\![3, n_{max} - 1]\!]$. Again this pattern may not be the pattern which maximizes the revenue because of the same things that we can do as previously with e_n, that is to say change a bit the values of the different b_is, for instance improve a bit the values of the bids following the constraints with α and decrease the values of the other bids. This is why for the worse revenue we prefer to employ the numerical approach and compare the results with the different patterns that we determined.

3.5 Conclusion

3.5.1 $R_{\%min}$ (Worse Case)

In general we find that the worse case is the Max inc_{max} Bid pattern, but it depends also on the value of n relative to the other parameters. Indeed:

- for $n = n_{max}$, the Min $step_{min}$ Bid Pattern is the best and unique pattern for the worst case and the revenue in percentage is:

$$R_{\%min} = 1 - \frac{b_1}{b_n}(inc_{max} + inc_{min}\alpha\frac{\alpha^{n-3} - 1}{\alpha - 1})$$

- for $3 \leq n \leq n_\Delta \leq n_{max}$ the Max inc_{max} Bid pattern is the best and $R_{\%min}$ is equal to:

$$R_{\%min} = 1 - inc_{max} \frac{b_1 + \frac{\Delta^{n-2}-1}{\alpha\Delta^{n+3}(\Delta-1)} b_n}{b_n}$$

- for $3 \leq n_\Delta < n < n_{max}$, again the Max inc_{max} Bid pattern is the best and $R_{\%min}$ is equal to::

$$R_{\%min} = 1 - \frac{b_1}{b_n}\left(inc_{max} + inc_{min}\alpha\left(\frac{\alpha^{n-j-3}-1}{\alpha-1}\right)\right) - \frac{inc_{max}}{\Delta^{j+1}}\left(\frac{\Delta^j-1}{\Delta-1}\right)$$

As explained in details in the section 3.4.1, the patterns found may give an global minimum, but we cannot prove it. Therefore we take a numerical approach in the section 3.7 to look for $R_{\%min}$.

3.5.2 $R_{\%max}$ (Best case)

We find that the best case for the revenue happens with the Min $step_{min}$ bidding pattern. In this case we have:

$$\forall n \in [\![3, n_{max}]\!], R_{\%max} = 1 - \frac{b_1}{b_n}(inc_{max} + inc_{min}\alpha\frac{\alpha^{n-3}-1}{\alpha-1})$$

Unlike $R_{\%min}$, the $R_{\%max}$ found with Theory is optimal. We are sure to get the maximal revenue with the formula written above.

3.5.3 Numerical Examples

In the tables below we show the couple ($R_{\%min}$,$R_{\%max}$) obtained with the formulas for multiple values of the ratio $\frac{b_n}{b_1}$ and multiples values of the number of bids n.
The pair (-1, -1) is shown when there is no solution, for $n > n_{max}$.

	n = 5	n = 10	n = 25	n = 50
$\frac{b_n}{b_1} = 7$	(89.05, 99.13)	(89.10, 98.68)	(88.64, 96.40)	(-1, -1)
$\frac{b_n}{b_1} = 10$	(89.30, 99.39)	(88.74, 99.07)	(88.46, 97.48)	(-1, -1)
$\frac{b_n}{b_1} = 20$	(89.55, 99.70)	(88.28, 99.54)	(88.16, 98.74)	(88.26, 95.07)
$\frac{b_n}{b_1} = 50$	(89.70, 99.88)	(87.98, 99.81)	(88.06, 99.50)	(87.83, 98.03)
$\frac{b_n}{b_1} = 70$	(89.73, 99.91)	(85.75, 99.87)	(87.98, 99.64)	(87.83, 98.59)
$\frac{b_n}{b_1} = 100$	(89.75, 99.94)	(85.77, 99.91)	(87.87, 99.75)	(87.84, 99.01)
$\frac{b_n}{b_1} = 500$	(89.79, 99.99)	(85.81, 99.98)	(87.82, 99.95)	(87.80, 99.80)
$\frac{b_n}{b_1} = 1000$	(89.80, 99.99)	(85.82, 99.99)	(87.81, 99.97)	(87.80, 99.90)

Table 1: $\boldsymbol{inc_{max} = 0.05}$, $\boldsymbol{inc_{min} = 0.005}$, **a = 0.112**, $\boldsymbol{step_{min} = 0.05}$

	n = 5	n = 10	n = 25	n = 50
$\frac{b_n}{b_1} = 7$	(86.22, 98.24)	(86.25, 97.05)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 10$	(85.69, 98.77)	(85.54, 97.94)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 20$	(85.36, 99.38)	(85.33, 98.97)	(85.79, 95.29)	(-1, -1)
$\frac{b_n}{b_1} = 50$	(83.93, 99.75)	(85.19, 99.59)	(84.88, 98.12)	(-1, -1)
$\frac{b_n}{b_1} = 70$	(83.99, 99.82)	(85.07, 99.71)	(85.26, 98.66)	(-1, -1)
$\frac{b_n}{b_1} = 100$	(84.03, 99.88)	(85.02, 99.79)	(84.99, 99.06)	(-1, -1)
$\frac{b_n}{b_1} = 500$	(84.11, 99.98)	(84.97, 99.96)	(84.94, 99.81)	(84.36, 97.70)
$\frac{b_n}{b_1} = 1000$	(84.12, 99.99)	(84.95, 99.98)	(84.95, 99.91)	(84.74, 98.85)

Table 2: $\boldsymbol{inc_{max} = 0.1}$, $\boldsymbol{inc_{min} = 0.01}$, **a = 0.112**, $\boldsymbol{step_{min} = 0.105}$

	n = 5	n = 10	n = 25	n = 50
$\frac{b_n}{b_1} = 7$	(82.42, 97.33)	(82.27, 95.13)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 10$	(81.75, 98.13)	(81.16, 96.59)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 20$	(81.75, 99.06)	(81.17, 98.30)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 50$	(81.30, 99.63)	(81.21, 99.32)	(80.25, 94.95)	(-1, -1)
$\frac{b_n}{b_1} = 70$	(81.28, 99.73)	(81.26, 99.51)	(81.61, 96.39)	(-1, -1)
$\frac{b_n}{b_1} = 100$	(81.27, 99.81)	(81.18, 99.66)	(80.62, 97.48)	(-1, -1)
$\frac{b_n}{b_1} = 500$	(79.76, 99.96)	(81.16, 99.93)	(81.08, 99.50)	(-1, -1)
$\frac{b_n}{b_1} = 1000$	(79.77, 99.98)	(81.16, 99.97)	(81.11, 99.75)	(-1, -1)

Table 3: $\boldsymbol{inc_{max} = 0.15}$, $\boldsymbol{inc_{min} = 0.015}$, **a = 0.112**, $\boldsymbol{step_{min} = 0.15}$

3.6 Minimising $R_{\%}$ whilst keep the auction solvent

We look for GBM auction parameters that minimise the revenue for the seller whilst keeping the auction solvent whatever the bidding pattern is. That is, we look for inc_{max}, inc_{min}, a, and $step_{min}$ so that the revenue in the worse case is 0 (i.e. $R_{\%min} = 0$), and the revenue in the best case $R_{\%max}$ is as low as possible.

Here we suppose that $(inc_{max}, inc_{min}, a, step_{min}) \in (\mathbb{R}_*^+)^4$. Let $n \in \mathbb{N}^*, n \leq n_{max}$. We set $inc_{min} = inc_{max} = \beta$. Then $\forall i \in [\![1, n-1]\!]$, $g_i = \beta$.
Then:

$$\begin{aligned} R_{\%} = 0 &\iff 1 - \frac{\sum_{i=1}^{n-1} g_i b_i}{b_n} = 0 \\ &\iff 1 - \beta \frac{\sum_{i=1}^{n-1} b_i}{b_n} = 0 \\ &\iff 1 - \beta \left(\frac{b_1}{b_n} + \frac{\sum_{i=2}^{n-1} b_i}{b_n} \right) = 0 \end{aligned}$$

The worst case happens when $\sum_{i=1}^{n-1} b_i$ is maximized, that is to say by taking the maximum value for the b_is. As $b_i \leq \frac{b_{i+1}}{\alpha}$the maximum value for each b_i is $\frac{b_{i+1}}{\alpha}$, thus we take this value. By immediate recursion, we get $b_i = \frac{b_n}{\alpha^{n-i}}$.

We replace in $R_{\%min}$:

$$\begin{aligned}
R_{\%min} &= 1 - \beta\left(\frac{b_1}{b_n} + \frac{\sum_{i=2}^{n-1}\alpha^{i-n}b_n}{b_n}\right)\\
&= 1 - \beta\left(\frac{b_1}{b_n} + \sum_{i=2}^{n-1}\alpha^{i-n}\right)\\
&= 1 - \beta\left(\frac{b_1}{b_n} + \frac{1}{\alpha^{n-2}}\sum_{i=0}^{n-3}\alpha^{i}\right)\\
&= 1 - \beta\left(\frac{b_1}{b_n} + \frac{\alpha^{n-2}-1}{\alpha^{n-2}(\alpha-1)}\right)\\
&= \delta - \beta\frac{\alpha^{n-2}-1}{\alpha^{n-2}(\alpha-1)}
\end{aligned}$$

with $\delta = 1 - \beta\frac{b_1}{b_n}$.

Now we rewrite the equation $R_{\%min} = 0$:

$$\begin{aligned}
R_{\%min} = 0 &\iff \beta\frac{\alpha^{n-2}-1}{\alpha^{n-2}(\alpha-1)} = \delta\\
&\iff \beta(\alpha^{n-2}-1) = \delta(\alpha^{n-2}(\alpha-1))\\
&\iff \delta\alpha^{n-1} - (\delta+\beta)\alpha^{n-2} + \beta = 0\\
&\iff f_n(\alpha) = 0
\end{aligned}$$

p with $f_n : x \longrightarrow \delta x^{n-1} - (\beta+\delta)x^{n-1} + \beta$.
$f_n(\alpha)$ is the revenue in percentage for a given n and α.

f_n is a continuous and differentiable function over $\mathbb{R}$ and $f_n'(x) = \delta(n-1)x^{n-2} - (\delta+\beta)(n-2)x^{n-3}$. We have a look at the sign of the derivative f_n', to know more about the monotonicity of f_n :

$$\begin{aligned}
f_n'(x) \geq 0 &\iff \delta(n-1)x^{n-2} - (\delta+\beta)(n-2)x^{n-3} \geq 0\\
&\iff \delta(n-1)x^{n-2} \geq (\delta+\beta)(n-2)x^{n-3}\\
&\iff x \geq \frac{(\delta+\beta)(n-2)}{\delta(n-1)} = \gamma
\end{aligned}$$

Below we present the table of variations concerning f_n:

x	0		1		γ		∞
$f'_n(x)$		$-$		$-$	0	$+$	$+$
$f_n(x)$	β	$\searrow$	0	$\searrow$	$f_n(\gamma)$	$\nearrow$	∞

We note that $f_n(1) = 0$: $f_n(1) = \delta \times 1 - (\beta + \delta) \times 1 + \beta = 0$. Thus $f_n(\gamma) < 0 \iff \gamma > 1$:

$$\begin{aligned}
\gamma > 1 &\iff \frac{(\delta+\beta)(n-2)}{\delta(n-1)} > 1 \\
&\iff 1 + \frac{\beta}{\delta} > \frac{n-1}{n-2} = \frac{1}{n-2} \\
&\iff \frac{\beta}{1 - \frac{b_1}{b_n}\beta} > \frac{1}{n-2} \\
&\iff \beta > (1 - \frac{b_1}{b_n}\beta)\frac{1}{n-2} \\
&\iff \beta > \frac{1}{(n-2)(1 + \frac{b_1}{(n-2)b_n})} \\
&\iff \beta > \frac{1}{(n-2) + \frac{b_1}{b_n}}
\end{aligned}$$

We have also :
$\lim\limits_{x \to \infty} f_n(x) = \infty$, thus there exists a $x^* \in (\gamma, \infty)$ so that $f_n(x^*) = 0$ (we will determine x^* numerically).

Let $n \in \mathbb{N}$.
Now let $\alpha \in (1, +\infty)$. Let us study the sequence $(f_n(\alpha))_{n \in \mathbb{N}}$.

$$\begin{aligned}
f_{n+1}(\alpha) - f_n(\alpha) &= \delta(\alpha^n - \alpha^{n-1}) - (\delta + \beta)(\alpha^{n-1} - \alpha^{n-2}) \\
&= \alpha^{n-2}[\delta(\alpha^2 - \alpha) - (\delta + \beta)(\alpha - 1)] \\
&= \alpha^{n-2}(\alpha - 1)[\delta\alpha - (\delta + \beta)]
\end{aligned}$$

As $\alpha \in (1, +\infty)$, $\alpha^{n-2}(\alpha - 1) > 0$, thus the monotonicity of $(f_n(\alpha))_{n \in \mathbb{N}}$ depends only on the term $\delta\alpha - (\delta + \beta)$.
In order to get a decreasing sequence we have to choose α such that $\delta\alpha - (\delta+\beta) < 0 \iff \alpha < 1 + \frac{\beta}{\delta}$.

To summarize, we need to choose β and α such that :

$$\beta > \frac{1}{(n-2)+\frac{b_1}{b_n}},\ \exists\, \widetilde{m} \in [\![n, n_{max}]\!],\ f_{\widetilde{m}}(\alpha) = 0,\ \forall m \in [\![n, n_{max}]\!], f_m(\alpha) \geq 0$$

The condition concerning the monotonicity of $(f_n(\alpha))_{n\in\mathbb{N}}$ depends only on β and not on m. One solution will be to choose $\beta = \frac{1}{(m-2)+\frac{b_1}{b_n}} + \frac{1}{n}$ and compare α with $1+\frac{\beta}{\delta}$. If $\alpha > 1+\frac{\beta}{\delta}$, then the sequence $(f_n(\alpha))_{n\in\mathbb{N}}$ is strictly non decreasing and we can choose m = n. Otherwise, the sequence $(f_n(\alpha))_{n\in\mathbb{N}}$ is decreasing and we need to choose $m = n_{max}$. By doing this, we will get $f_{n_{max}}(\alpha) = 0$, and $\forall m \in [\![n, n_{max}-1]\!]$, $f_m(\alpha) > 0$ that is to say in the worst case the revenue will be either 0 % (when we will have n_{max} bids), or strictly positive.

If we choose to get a decreasing sequence, the problem becomes a bit more complicated. Indeed we need to find α and β such that $f_{n_{max}}(\alpha) = 0$ but n_{max} depends on α (reminder $n_{max} = \frac{log(b_n)-log(b_1)}{log(\alpha)}$).
We chose a decreasing sequence for $(f_n(\alpha))_{n\in\mathbb{N}}$ in order to get a revenue of 0 % for a large number of bids.
In this case we need to find β and $1 + step_{min}$ so that:

$$\beta > \frac{1}{(n-2)+\frac{b_1}{b_n}},\ 1 + step_{min} < 1 + \frac{\beta}{1-\beta\frac{b_1}{b_n}},\ f_{n_{max}}(1+step_{min}) = 0,$$

$$\forall m \in [\![n, n_{max}]\!], f_m(1+step_{min}) \geq 0$$

The problem can be regarded as an optimization problem with three non linear constraints and one equality constraint ($f_{n_{max}}(\alpha) = 0$).

For instance with $\frac{b_n}{b_1} = 100$, we find the following presets:

$$(inc_{max} \simeq 0.0995, inc_{min} \simeq 0.0995, a, step_{min} \simeq 0.0984)$$

We find $n_{max} = 49$ and the following percentages:

$$\begin{aligned} f_{n_{max}=49}(\alpha) &= 0\,\% \\ f_{48}(\alpha) &\simeq 0.89\,\% \\ &\vdots \\ f_{20}(\alpha) &\simeq 9.30\,\% \\ &\vdots \\ f_{10}(\alpha) &\simeq 9.70\,\%, \\ &\vdots \\ f_{3}(\alpha) &\simeq 9.81\,\% \end{aligned}$$

With the presets that we found we can also have a look at the revenue in percentage in the best case.
We can use the formula determined in the section 3.4.1 to get the revenue in the best case.
We have the following results:

$$\begin{aligned} R_{\%max,\, n_{max}=49} &\simeq 17.64\,\% \\ R_{\%max,\, 48} &\simeq 25.11\,\% \\ &\vdots \\ R_{\%max,\, 20} &\simeq 95.53\,\% \\ &\vdots \\ R_{\%max,\, 10} &\simeq 98.87\,\%, \\ &\vdots \\ R_{\%max,\, 3} &\simeq 99.79\,\% \end{aligned}$$

3.7 Numerical Approach

We attempted to solve the optimization problem of finding the minimum and maximum values of the revenue thought a numerical approach using Python. We searched for the minimum at local level. At the global level, we found issues with the global algorithms and decided to not pursue this method further. The results with local algorithms were satisfactory.

Some issues appeared with the optimization package in Python. Indeed, most of the algorithms do not work because they cannot handle the constraints of our problem. Some algorithms that were able to manage these constraints (e.g. SLSQP and Cobyla) did not seem to work as they return as solution (optimal value) the starting point. However, we made some progress with the algorithm "trust-constr" [3], which gives good results.

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[3] see here: `https://docs.scipy.org/doc/scipy/reference/generated/scipy.optimize.minimize.html`

In the table below we show the results obtained with this numerical approach for different presets (see below) and multiple values of $\frac{b_n}{b_1}$

	n = 5	n = 10	n = 25	n = 50
$\frac{b_n}{b_1} = 7$	(88.05, 98.95)	(88.22, 98.46)	(89.06, 95.95)	(-1, -1)
$\frac{b_n}{b_1} = 10$	(88.03, 99.27)	(87.36, 98.92)	(88.69, 97.16)	(-1, -1)
$\frac{b_n}{b_1} = 20$	(88.11, 99.64)	(86.59, 99.46)	(88.27, 98.57)	(88.68, 94.54)
$\frac{b_n}{b_1} = 50$	(88.27, 99.85)	(85.55, 99.78)	(87.74, 99.43)	(87.9, 97.81)
$\frac{b_n}{b_1} = 70$	(88.26, 99.9)	(85.78, 99.84)	(86.85, 99.59)	(87.33, 98.44)
$\frac{b_n}{b_1} = 100$	(88.27, 99.93)	(85.93, 99.89)	(87.17, 99.7)	(87.49, 98.9)
$\frac{b_n}{b_1} = 500$	(88.3, 99.99)	(85.8, 99.98)	(87.31, 99.94)	(86.38, 99.78)
$\frac{b_n}{b_1} = 1000$	(88.31, 99.99)	(85.77, 99.99)	(86.63, 99.97)	(86.07, 99.89)

Table 4: **$inc_{max} = 0.05$, $inc_{min} = 0.005$, a = 0.112, $step_{min} = 0.05$**

	n = 5	n = 10	n = 25	n = 50
$\frac{b_n}{b_1} = 7$	(85.32, 97.84)	(86.08, 96.38)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 10$	(83.25, 98.49)	(85.84, 97.47)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 20$	(82.57, 99.24)	(85.26, 98.73)	(87.38, 94.25)	(-1, -1)
$\frac{b_n}{b_1} = 50$	(82.89, 99.7)	(83.3, 99.49)	(86.01, 97.70)	(-1, -1)
$\frac{b_n}{b_1} = 70$	(82.90, 99.78)	(82.52, 99.63)	(85.57, 98.35)	(-1, -1)
$\frac{b_n}{b_1} = 100$	(84.19, 99.85)	(82.55, 99.75)	(84.59, 98.85)	(-1, -1)
$\frac{b_n}{b_1} = 500$	(83.02 99.97)	(81.98, 99.95)	(82.47, 99.77)	(88.22, 97.19)
$\frac{b_n}{b_1} = 1000$	(83.02, 99.98)	(82.03, 99.97)	(82.49, 99.88)	(82.97, 98.6)

Table 5: **$inc_{max} = 0.1$, $inc_{min} = 0.01$, a = 0.112, $step_{min} = 0.105$**

	n = 5	n = 10	n = 25	n = 50
$\frac{b_n}{b_1} = 7$	(82.3, 96.62)	(86.06, 93.71)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 10$	(82.24, 97.63)	(82.65, 95.6)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 20$	(79.74, 98.81)	(82.57, 97.79)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 50$	(78.69, 99.52)	(82.20, 99.11)	(87.85, 93.34)	(-1, -1)
$\frac{b_n}{b_1} = 70$	(78.77, 99.66)	(81.01, 99.37)	(83.59, 95.24)	(-1, -1)
$\frac{b_n}{b_1} = 100$	(78.91, 99.76)	(80.15, 99.55)	(83.68, 96.67)	(-1, -1)
$\frac{b_n}{b_1} = 500$	(78.96, 99.95)	(86.96, 99.91)	(80.18, 99.33)	(-1, -1)
$\frac{b_n}{b_1} = 1000$	(78.97, 99.98)	(79.65, 99.96)	(78.86, 99.67)	(-1, -1)

Table 6: $\boldsymbol{inc_{max} = 0.15, inc_{min} = 0.015, \mathbf{a} = 0.112, step_{min} = 0.15}$

	n = 5	n = 10	n = 25	n = 50
$\frac{b_n}{b_1} = 7$	(-1, -1)	(-1, -1)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 10$	(77.64, 82.17)	(-1, -1)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 20$	(77.53, 91.03)	(-1, -1)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 50$	(63.49, 96.44)	(-1, -1)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 70$	(63.5, 97.45)	(79.66, 83.15)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 100$	(63.65, 98.22)	(79.75, 88.2)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 500$	(62.04, 99.64)	(63.81, 97.64)	(-1, -1)	(-1, -1)
$\frac{b_n}{b_1} = 1000$	(62.84, 99.82)	(63.32, 98.82)	(-1, -1)	(-1, -1)

Table 7: $\boldsymbol{inc_{max} = 0.5, inc_{min} = 0.1, \mathbf{a} = 0.112, step_{min} = 0.5}$

Now let us compare the results got numerically with the patterns that we found in the section 3.4.

3.8 Comparisons between the formulas derived from the patterns and the numerical resolution

Here we compare the pairs ($R_{\%min}$, $R_{\%max}$) we obtained with the pattern analysis approach and with the numerical resolution for different values of $\frac{b_n}{b_1}$ and n. The pairs in the tables below represent the difference between the theoretical percentages and the numerical percentages.

	n = 5	n = 10	n = 25	n = 50
$\frac{b_n}{b_1} = 7$	(1.00, 0.18)	(0.88, 0.22)	(-0.42, 0.45)	(0, 0)
$\frac{b_n}{b_1} = 10$	(1.27, 0.12)	(1.38, 0.15)	(-0.23, 0.32)	(0, 0)
$\frac{b_n}{b_1} = 20$	(1.44, 0.06)	(1.69, 0.08)	(-0.11, 0.17)	(-0.42, 0.53)
$\frac{b_n}{b_1} = 50$	(1.43, 0.03)	(2.43, 0.03)	(0.32, 0.07)	(-0.07, 0.22)
$\frac{b_n}{b_1} = 70$	(1.47, 0.01)	(-0.03, 0.03)	(1.13, 0.05)	(0.50, 0.15)
$\frac{b_n}{b_1} = 100$	(1.48, 0.01)	(-0.16, 0.02)	(0.70, 0.05)	(0.35, 0.11)
$\frac{b_n}{b_1} = 500$	(1.49, -0.00)	(0.01, 0.00)	(0.51, 0.01)	(1.42, 0.02)
$\frac{b_n}{b_1} = 1000$	(1.49, 0.00)	(0.05, 0.00)	(1.18, 0.00)	(1.73, 0.01)

Table 8: $\boldsymbol{inc_{max} = 0.05}$, $\boldsymbol{inc_{min} = 0.005}$, **a = 0.112**, $\boldsymbol{step_{min} = 0.05}$

	n = 5	n = 10	n = 25	n = 50
$\frac{b_n}{b_1} = 7$	(0.66, 0.40)	(0.19, 0.66)	(0, 0)	(0, 0)
$\frac{b_n}{b_1} = 10$	(2.50, 0.28)	(-0.27, 0.47)	(0, 0)	(0, 0)
$\frac{b_n}{b_1} = 20$	(2.66, 0.14)	(0.12, 0.24)	(-1.54, 1.04)	(0, 0)
$\frac{b_n}{b_1} = 50$	(1.08, 0.05)	(1.94, 0.10)	(-1.10, 0.42)	(0, 0)
$\frac{b_n}{b_1} = 70$	(1.12, 0.04)	(2.50, 0.07)	(0.28, 0.30)	(0, 0)
$\frac{b_n}{b_1} = 100$	(1.07, 0.03)	(2.53, 0.04)	(-0.47, 0.21)	(0, 0)
$\frac{b_n}{b_1} = 500$	(1.13, 0.01)	(2.12, 0.01)	(2.59, 0.04)	(-0.39, 0.51)
$\frac{b_n}{b_1} = 1000$	(1.14, 0.01)	(2.97, 0.01)	(2.87, 0.03)	(1.82, 0.25)

Table 9: $\boldsymbol{inc_{max} = 0.1}$, $\boldsymbol{inc_{min} = 0.01}$, **a = 0.112**, $\boldsymbol{step_{min} = 0.105}$

	n = 5	n = 10	n = 25	n = 50
$\frac{b_n}{b_1} = 7$	(0.12, 0.71)	(-3.79, 1.42)	(0, 0)	(0, 0)
$\frac{b_n}{b_1} = 10$	(-0.49, 0.50)	(-1.49, 0.99)	(0, 0)	(0, 0)
$\frac{b_n}{b_1} = 20$	(2.01, 0.25)	(-1.40, 0.51)	(0, 0)	(0, 0)
$\frac{b_n}{b_1} = 50$	(2.61, 0.11)	(-0.99, 0.21)	(-7.60, 1.61)	(0, 0)
$\frac{b_n}{b_1} = 70$	(2.51, 0.07)	(0.25, 0.14)	(-1.98, 1.15)	(0, 0)
$\frac{b_n}{b_1} = 100$	(2.36, 0.05)	(1.03, 0.11)	(-3.06, 0.81)	(0, 0)
$\frac{b_n}{b_1} = 500$	(0.80, 0.01)	(-5.80, 0.02)	(0.90, 0.17)	(0, 0)
$\frac{b_n}{b_1} = 1000$	(0.80, 0.00)	(1.51, 0.01)	(2.25, 0.08)	(0, 0)

Table 10: $\boldsymbol{inc_{max} = 0.15}$, $\boldsymbol{inc_{min} = 0.015}$, **a = 0.112**, $\boldsymbol{step_{min} = 0.15}$

The difference in maximum revenue between the numerical and theoretical approaches are small, which we can attribute to the numerical approximations of the algorithms (e.g. rounding errors).

The difference in minimum revenue between the numerical and theoretical approaches are mostly negative, which would make us think that the numerical approach finds patterns where R_min is lower than what the theoretical approach finds. However, there are also in a few cases considerable differences between the numerical and theoretical approaches for the minimum revenue. For instance in Table 10, for $\frac{b_n}{b_1} = 50$ and $n = 25$ we find a difference of 7.60%. These differences may come from the fact that the algorithm used for the numerical approach works only at a local level, it computes only a local minimum revenue, whereas with the theoretical approach all the work is done at a global level.

4 GBM in Auction Theory

4.1 A Generalisation of the English Auction

We do a reminder of English auction:

The auctioneer opens the auction by announcing a suggested opening bid, a starting price or reserve for the item on sale.

Then, the auctioneer accepts increasingly higher bids from the floor, consisting of buyers with an interest in the item. The auctioneer usually determines the minimum increment of bids, often raising it when bidding goes high.

The highest bidder at any given moment is considered to have the standing bid, which can only be displaced by a higher bid from a competing buyer.

If no competing bidder challenges the standing bid within a given time frame, the standing bid becomes the winner, and the item is sold to the highest bidder at a price equal to their bid.

The GBM auction is a generalization of the English auction, as an English auction is a GBM auction when $inc_{max} = step_{min} = 0$

Proof:
if $inc_{max} = 0$ then for every (x, y) in $\mathbb{R}^2_+$, $g(x, y) = 0$, that is to say the revenue of the auction is the highest bid (no incentives are distributed). Also, $step_{min} = 0$ implies that any new bid must simply be higher than the standing bid, without setting a minimum increment (unlike the GBM which requires new bids to be at least equal to $\alpha \times$ standing bid).
Thus a GBM auction with $inc_{max} = step_{min} = 0$ is an English auction.

Because of this, it is likely that the GBM auction challenges or falls outside of the scope of several theorems and results in auction theory. Our objective is to review the key findings in auction theory literature, study how the GBM auction fits, and if new findings can be found.

4.2 Expected Value

Let's analyse an auction that follows the benchmark model as defined by McAfee and McMillan[2], except that bidders are not necessarily risk-neutral.

We assume that:

- The bidders seek to maximise the expected value of a von Neumann–Morgenstern utility function[4]
- Each bidder has a private valuation for the item independently drawn from some probability distribution
- The bidders possess symmetric information
- The bidder's payoff is represented as a function of only the bids

We consider a bidder with a private valuation v, and placing a bid of value $b(v)$. The bid is a strictly non decreasing function of the valuation v, it is common to all the bidders, it is called the common bid function:

$$\begin{array}{rccc} b: & [0,V] & \longrightarrow & \mathbb{R}_+ \\ & v & \longmapsto & b(v) \end{array}$$

We suppose that the function b is bijective, strictly non decreasing. Then:

$$\begin{aligned} P(b_j \leq b) &= P(b(v_j) \leq b(v)) \\ &= P(v_j \leq b^{-1}(b(v))) \\ &= P(v_j \leq v) \\ &= F(b^{-1}(b(v))) \text{ or } F(v) \end{aligned}$$

For better clarity, we omit the "(v)" when we talk about "$b(v)$".
In the case of an English auction, the bidder's payoff is

$$u = \begin{cases} U(0) & \textit{if lose} \\ U(v-b) & \textit{if win} \end{cases}$$

---

[4] The von Neumann–Morgenstern utility theorem shows that, under certain axioms of rational behavior, a decision-maker faced with risky (probabilistic) outcomes of different choices will behave as if he or she is maximizing the expected value of a function defined over the potential outcomes at some specified point in the future.

where U is an increasing, concave von Neumann-Morgenstern utility function. The bidder's problem is to maximise the expected value $E[X = b] = \mathrm{P}(win|b)U(v-b) + \mathrm{P}(lose|b)U(0)$, with $\mathrm{P}(win|b)$ and $\mathrm{P}(lose|b)$ the probability to win and to lose the auction with that bid, respectively.

In the case of a GBM function of incentive function g, the bidder's payoff becomes

$$u = \begin{cases} U(b \times g(b)) & \textit{if lose} \\ U(v-b) & \textit{if win} \end{cases}$$

where $b \times g(b)$ is the incentive amount the bidder will receive if outbid. The bidder's problem become to maximise the expected value $E_{GBM}[X = b] = \mathrm{P}(win|b)U(v-b) + \mathrm{P}(lose|b)U(b \times g(b))$.

We assume a bidder is willing to place a bid b on an item in an English auction. That is, the bidder's private valuation t and his attitude to risk defined by the utility function U are so that the value of the expected value E is high enough to make the decision to place the bid b, and by definition any higher expected value would result in the bidder also placing that bid.

$$\begin{aligned} \{win|b\} &= \{\text{the bid b is higher than the bids of the other bidders}\} \\ &= \{b_j \leq b \, \forall j \in [\![1,n]\!] \backslash \{i\}\} \\ &= \{b_j \leq b\} \cap \ldots \cap \{b_n \leq b\} \end{aligned}$$

Thus:

$$P(win|b) = \prod_{j=1, j\neq i}^{n} P(b_j \leq b) = \prod_{j=1, j\neq i}^{n} P(v_j \leq v) = F(v)^{n-1}$$

Assuming that the private value distribution stays the same for a GBM auction of the same item, the probability of winning or losing the auction for the same bid b and private valuation v are equal (NOW IT IS OK). Therefore, we have:

$$\begin{aligned} b \times g(b) > 0 &\iff U(b \times g(b)) > U(0) \text{ because } U \text{ is increasing} \\ &\iff \mathrm{P}(lose|b)U(b \times g(b)) > \mathrm{P}(lose|b)U(0) \\ &\iff E_{GBM}[X = b] > E[X = b] \text{ by adding } \mathrm{P}(win|b)U(v-b) \end{aligned}$$

As a result, with all parameters being equal, the GBM auction is preferred to the English auction by the bidder.
The preference goes further still, because g is increasing thus the incentive amount $b \times g(b)$ increases equally (if inc_{max} has been reached) or faster (if inc_{max} has not been reached) than b.

We assume the expected value E_1 of a bidder willing to place a bid b_1 on an item in an English auction, and the expected value $E_{2,GBM}$ of that same bidder willing to place a bid b_2 on the same item in a GBM auction, so that $E_1 = E_{2,GBM}$. Let's try and demonstrate that $b_2 > b_1$: (TO MODIFY)

$$\begin{aligned}
&E_1 = E_{2,GBM} \\
\iff &\mathrm{P}(win|b_1)U(v-b_1) + \mathrm{P}(lose|b_1)U(0) \\
&\quad = \mathrm{P}(win|b_2)U(v-b_2) + \mathrm{P}(lose|b_2)U(g(b_2)b_2)) \\
\iff &\mathrm{P}(win|b_1)U(v-b_1) - \mathrm{P}(win|b_2)U(v-b_2) \\
&\quad = \mathrm{P}(lose|b_2)U(g(b_2)b_2) - \mathrm{P}(lose|b_1)U(0)
\end{aligned}$$

$\mathrm{P}(win|b_1) = F(v)^{n-1}$.The same holds for $\mathrm{P}(win|b_2)$. Thus $\mathrm{P}(win|b_1) = \mathrm{P}(win|b_2)$. We denote them by $\mathrm{P}(win|b)$.
Then the equality $E_1 = E_{2,GBM}$ becomes:

$$\begin{aligned}
&E_1 = E_{2,GBM} \\
\iff\ &\mathrm{P}(win|b)U(v-b_1) - \mathrm{P}(win|b)U(v-b_2) = \mathrm{P}(lose|b)U(g(b_2)b_2) - \mathrm{P}(lose|b)U(0) \\
\iff\ &\mathrm{P}(win|b)[U(v-b_1) - U(v-b_2)] = \mathrm{P}(lose|b)[U(g(b_2)b_2) - U(0)]
\end{aligned}$$

$g(b_2)b_2 \geq 0 \implies U(g(b_2)b_2) - U(0) \geq 0$ because U is non decreasing.
As $P(win|b) \geq 0$ and $P(lose|b) \geq 0$ we get that $U(t-b_1) - U(t-b_2) \geq 0$.

$$\begin{aligned}
U(v-b_1) - U(v-b_2) \geq 0 &\iff v - b_1 - (v - b_2) \geq 0 \text{ (U non decreasing)} \\
&\iff -b_1 \geq -b_2 \\
&\iff b_2 \geq b_1 \, (QED)
\end{aligned}$$

4.3 Dominant Strategy

Let's look for the dominant strategy in a GBM auction. That is, the course of action that results in the highest payoff for the bidder. We represent the GBM auction as a set of n bids, with the i^{th} bidder placing a bid of amount b_i, $i \in [\![1, n]\!]$, with a starting bid of $b_1 > 0$. We make the following assumptions:

- Bidders are risk neutral[5] in evaluating their payoffs under uncertainty. That is, each bidder seeks merely to maximize the mathematical expectation of his payoff.
- Private Values Model: each bidder knows his own valuation of the object, but does not know the valuation of the other bidders. That is to say the i^{th} bidder payoff depends on the j^{th} bidder's valuation only through that bidder's action.

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[5] Risk neutral is a concept used in both game theory studies and in finance. It refers to a mindset where an individual is indifferent to risk when making an investment decision.
This mindset is not derived from calculation or rational deduction, but rather from an emotional preference.
A person with a risk-neutral approach simply doesn't focus on the risk–regardless of whether or not that is an ill-advised thing to do. This mindset is often situational and can be dependent on price or other external factors.

- Independence of the private information. That is, the joint distribution function, $F(v_1, ..., v_n)$, is given by the product of separate distribution functions, $F_i(.)$, for each v_i.
- The bidders are symmetric. They will all be using the same bidding function F. Thus $F_i = F$. In other words, the assumption that the various v_is are identically distributed, as well as independent, random variables.
- Payments to bidders are a function of bids alone.
- The GBM auction is used to sell only one object.

In the following subsections, we present the most important theorems and the auction theory concepts, and study how the GBM auction model fits.
Let $n \in \mathbb{N}^*$ the number of bidders. We denote them by $1, \ldots, n$. In general, with the GBM auction we will have a few bids ($n \leq 60$).
Before the auction starts, each bidder has a valuation of the object, that we can denote by v_i, $i \in [\![1, n]\!]$. v_i can be seen as a random variable and drawn from a known joint distribution $F(v_1, ..., v_n)$.
Bidder i's payoff : $= v_i - b_i$ if he wins, $g(b_{i-1}, b_i)b_i$ if he loses(he earns some money). Therefore the expected revenue of the bidder i is

$$EU_i = P_{win}(v_i - b_i) + P_{lose}g(b_{i-1}, b_i)b_i$$

Let us try to rewrite the probability that the bidder i wins the auction:

$$\begin{aligned} P_{win}(Bidder_i) &= P(\{v_1 < v_i\} \cap \ldots \cap \{v_{i-1} < v_i\} \cap \{v_{i+1} < v_i\} \cap \ldots \cap \{v_n < v_i\}) \\ &= \prod_{j=1, j\neq i}^{n} P(\{v_j < v_i\}) \text{ (Independence)} \\ &= \prod_{j=1, j\neq i}^{n} F(v_i) \\ &= F(v_i)^{n-1} \end{aligned}$$

We notice that $P_{lose} = 1 - P_{win} = 1 - F(v_i)^{n-1}$

We are looking for $\underset{x_i}{max}\ EU_i = \underset{x_i}{max}\ F(s^{-1}(x_i))^{n-1}(v_i - x_i) + (1 - F(s^{-1}(x_i))^{n-1})g_i x_i$
For readability of the computations, we will replace $s^{-1}(x_i)$ by s^{-1}, $F(s^{-1}(x_i))^{n-1}$ by F^{n-1}, $s(v_i)$ by s and $s'(v_i)$ by s'.

In order to get the max of EU_i, we can have a look at the derivative:

$$\begin{aligned} \frac{dEU_i}{dx_i} &= -F^{n-1} + (v_i - x_i)(n-1)F^{n-2}F'\frac{ds^{-1}(x_i)}{dx_i} \\ &\quad + (1 - F^{n-1})(g_i' x_i + g_i) - (n-1)F^{n-2}F'\frac{ds^{-1}(x_i)}{dx_i} g_i x_i \end{aligned}$$

From our lectures concerning derivative we know that :

- $\frac{ds^{-1}(x_i)}{dx_i} = \frac{1}{s'(s^{-1}(x_i))} = \frac{1}{s'(v_i)} = \frac{1}{s'}$
- As F is a distribution F' is equal to its density f
- $g_i' = \begin{cases} 0 \text{ if } x_i \geq \Delta b_{i-1} \\ \frac{a}{\alpha b_{i-1}} \text{ otherwise} \end{cases}$
- $g_i' x_i + g_i = \begin{cases} inc_{max} \text{ if } x_i \geq \Delta b_{i-1} \\ \frac{2ax_i}{\alpha b_{i-1}} + inc_{min} - a \text{ otherwise} \end{cases}$

With this we can rewrite the quantity $\frac{dEU_i}{dx_i}$:

$$\begin{aligned} & -F^{n-1} + (v_i - x_i)(n-1)F^{n-2}F'\frac{ds^{-1}(x_i)}{dx_i} + (1-F^{n-1})(g_i'x_i + g_i) \\ & -(n-1)F^{n-2}F'\frac{ds^{-1}(x_i)}{dx_i}g_i x_i \\ & = -F^{n-1} + (v_i - x_i)(n-1)F^{n-2}f\frac{1}{s'} + (1-F^{n-1})(g_i'x_i + g_i) \\ & -(n-1)F^{n-2}f\frac{1}{s'}g_i x_i \end{aligned}$$

We suppose that $x_i \geq \Delta b_{i-1}$.Then all the terms with g_i' disappear. We denote by I_m the quantity $1 + inc_{max}$

$$\begin{aligned} \frac{dEU_i}{dx_i} = 0 \iff & -F^{n-1} + (v_i - x_i)(n-1)F^{n-2}f\frac{1}{s'} + (1-F^{n-1})g_i \\ & -(n-1)F^{n-2}f\frac{1}{s'}g_i x_i = 0 \\ \iff & -F^{n-1}s' + (v_i - x_i)(n-1)F^{n-2}f + s'(1-F^{n-1})inc_{max} \\ & -(n-1)F^{n-2}finc_{max}x_i = 0 \\ \iff & (n-1)I_mF^{n-2}fx_i - inc_{max}s' + s'I_mF^{n-1} = (n-1)F^{n-2}fv_i \end{aligned}$$

But $x_i = s(v_i)$ thus:

$$\begin{aligned} & (n-1)I_mF^{n-2}fx_i - inc_{max}s' + s'I_mF^{n-1} = (n-1)F^{n-2}fv_i \\ \iff & (n-1)I_mF^{n-2}fs + (I_mF^{n-1} - inc_{max})s' = (n-1)F^{n-2}fv_i \end{aligned}$$

The left hand side is the derivative of $(I_mF^{n-1} - inc_{max})s$ with respect to v_i:

$$\begin{aligned} \frac{d(I_mF^{n-1} - inc_{max})s}{dv_i} & = \frac{d(I_mF^{n-1} - inc_{max})}{dv_i}s + (I_mF^{n-1} - inc_{max})s' \\ & = (n-1)I_mF^{n-2}fs + (I_mF^{n-1} - inc_{max})s' \end{aligned}$$

Thus (for the moment we suppose that s(r) = 0):

$$\begin{aligned}
(n-1)I_mF^{n-2}fs + (I_mF^{n-1} - inc_{max})s' &= (n-1)F^{n-2}fv_i \\
\iff \frac{d(I_mF^{n-1} - inc_{max})s}{dv_i} &= (n-1)F^{n-2}fv_i \\
\implies (I_mF^{n-1} - inc_{max})s &= \int_r^{v_i} (n-1)F(v)^{n-2}f(v)v\,\mathrm{d}v \\
&= [F^{n-1}(v)v]_r^{v_i} - \int_r^{v_i} F(v)^{n-1}\,\mathrm{d}v \\
&\text{(Integration by parts)}
\end{aligned}$$

A similar reasoning for $x_i < \Delta b_{i-1}$ yields:

$$(n-1)JF^{n-2}fs^2 + 2J(F^{n-1}-1)ss' + (n-1)IF^{n-2}fs + (F^{n-1}I - I + 1)s' = (n-1)F^{n-2}fv_i$$

with $I = 2 + inc_{min} - a$, $J = \frac{a}{\alpha b_{i-1}}$
Again, we remark that the left hand side is the derivative of $J(F^{n-1}-1)s^2 + (IF^{n-1} - I + 1)s$ which implies:

$$J(F^{n-1}-1)s^2 + (IF^{n-1} - I + 1)s = \int_r^{v_i} (n-1)F(v)^{n-2}f(v)v\,\mathrm{d}v$$

For instance if F is the uniform distribution in [0, 1], $F(v_i)^{n-1} = v_i^{n-1}$.
We suppose that $r = 0$ and $s(r) = 0$. If $x \geq \Delta b_{i-1}$:

$$\begin{aligned}
(I_mF^{n-1}(v_i) - inc_{max})s(v_i) &= [F^{n-1}(v)v]_r^{v_i} - \int_r^{v_i} F(v)^{n-1}\,\mathrm{d}v \\
\iff (I_mv_i^{n-1} - inc_{max})s(v_i) &= v_i^n - \int_0^{v_i} v^{n-1}\,\mathrm{d}v \\
\iff (I_mv_i^{n-1} - inc_{max})s(v_i) &= v_i^n - \frac{v_i^n}{n} \\
\iff s(v_i) &= \frac{v_i^n - \frac{v_i^n}{n}}{I_mv_i^{n-1} - inc_{max}} \\
\iff s(v_i) &= \frac{(n-1)v_i^n}{n[(1+inc_{max})v_i^{n-1} - inc_{max}]}
\end{aligned}$$

We denote by $s_n^{[1]}$ the maximum value of the $s(v_i)$, and we also split the terms $g(s(v_{i-1}, s(v_i))$. We gather all those that are equal to inc_{max} with the sum from $n_{\leq\Delta} + 1$ to $n-1$ and the remaining terms in the sum from 1 to $n_{\leq\Delta}$.

Results concerning Expected Revenue:

$$\begin{aligned}
\mathbb{E}(R) &= \mathbb{E}\left(s_n^{[1]} - \sum_{i=1}^{n-1} g(s(v_{i-1}, s(v_i))s(v_i)\right) \\
&= \mathbb{E}(s_n^{[1]}) - \sum_{i=1}^{n-1} \mathbb{E}(g(s(v_{i-1}, s(v_i))s(v_i)) \\
&= \mathbb{E}(s_n^{[1]}) - \sum_{i=1}^{n_{\leq\Delta}} \mathbb{E}(g(s(v_{i-1}, s(v_i))s(v_i)) - \sum_{i=n_{\leq\Delta}+1}^{n-1} E(g(s(v_{i-1}, s(v_i))s(v_i)) \\
&= \mathbb{E}(s_n^{[1]}) - \sum_{i=1}^{n_{\leq\Delta}} \mathbb{E}(g(s(v_{i-1}, s(v_i))s(v_i)) - \sum_{i=n_{\leq\Delta}+1}^{n-1} E(inc_{max} s(v_i)) \\
&= \int_0^1 \frac{(n-1)v^n}{n[(1+inc_{max})v^{n-1} - inc_{max}]} \mathrm{d}v - \sum_{i=1}^{n_{\leq\Delta}} \mathbb{E}(g(s(v_{i-1}, s(v_i))s(v_i)) \\
&- inc_{max} \sum_{i=n_{\leq\Delta}+1}^{n-1} \int_0^1 \frac{(n-1)v^n}{n[(1+inc_{max})v^{n-1} - inc_{max}]} \mathrm{d}v \\
&= \left(1 - inc_{max} \frac{(n-1)(n-n_{\leq\Delta}-1)}{n}\right) \int_0^1 \frac{v^n}{[(1+inc_{max})v^{n-1} - inc_{max}]} \mathrm{d}v \\
&- \sum_{i=1}^{n_{\leq\Delta}} \mathbb{E}(g(s(v_{i-1}, s(v_i))s(v_i))
\end{aligned}$$

With the assumptions of the IPV model, we were able to determine the function b (however we gave an explicit formula for it only when $g_i = inc_{max}$ for the sake of simplicity). But the obtained result is not acceptable, because it is possible to get a negative bid.

4.4 Revenue Equivalence

The Revenue Equivalence Theorem states that, under the following assumptions:

- We use the assumptions of the Independent Private Values Model (see the beginning of the previous section).
- We have symmetric bidders (private values are i.i.d. draws from a probability distribution F).
- Let us assume F has support [0,T].
- We consider any auction where, in equilibrium, we have both (1) the bidder with the highest value wins and (2) the expected payment from a bidder

with the lowest possible type is 0. [6]

Then the expected payoff to each bidder, and the seller's expected revenue, is the same across all such auctions that verify the above assumptions.

For the English auction, all these hypotheses are verified. We see if the GBM auction verifies the multiple assumptions.
The first three assumptions can be made for the GBM auction, as we have in the previous section.

Let's look at the fourth assumption:
By definition of the GBM auction, the bidder with the highest value does win. However, the expected payment from a bidder with the lowest possible type is not 0. Indeed, if a bidder has the lowest valuation, let us say v_0, he may place a bid greater than v_0 such that his payoff stays above 0. In other terms, that bidder has the possibility to win money with a bid greater than v_0.
Thus the Revenue Equivalence Theorem does not work for the GBM auction.

5 Other Properties and Variations

5.1 Sale of items with known value

The GBM auction is best suited for selling unique assets with no real benchmark value. Due to the required minimum step between each bid, which can be significant depending on the incentive parameters, if an item of clearly defined and known commercial value is sold through a GBM auction, it will theoretically generate less revenue than a traditional English auction (unless of course that auction has the same minimum step between each bid). In this regard, this is not a property specific to the GBM auction, however this problem is more likely to arise with a GBM auction because it is required that $step_{min} \geq inc_{max}$ for a GBM auction to always stay solvent, and the bigger inc_{max}, the bigger $step_{min}$.

5.2 Multi-unit GBM Auction

The GBM auction model can also be applied to a multi-unit discriminatory price auction i.e. selling multiple semi-homogeneous items at different prices in a single auction.

An example of a multi-unit discriminatory price auction would be the sale of a limited edition of 100 identical sport trading cards, numbered from 1 to 100. At the end of the auction, the top 100 bids win a trading card, with the highest bidder winning the trading card number 1, the second highest bidder winning the trading card number 2, et cetera.

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[6] i.e give a bidder with the lowest valuation zero profits (or $U(0) = 0$)

A multi-unit GBM auction of k units $u_1 ... u_k$ works as follows:

At any given moment, there are k standing bids. These k standing bids are ranked from highest to lowest. We call the 1^{st} standing bid (the bid in 1^{st} position) the top bid, and the k^{th} standing bid the bottom bid. If less than k bids have been placed, the unfilled positions (including the bottom bid) are equal to 0.

Any new bid must exceed the value of the bottom bid in order to become a standing bid. When a new standing bid is placed, all bids of a lower amount that the new bid get move down the standing bid order by 1, and the bottom bid is automatically displaced (i.e. it falls outside of the standing bid table). If that new bid becomes the top bid (i.e. it becomes the new highest bidder), then the previous highest bidder receives an incentive, and the new highest bidder will also receive an incentive if they are outbid. No other standing bids gets an incentive when outbid outside of the top bid. As a result of this incentive mechanism, there is a minimum step that a new bid must abide only in order to displace the previous highest bid.

The k highest bidders (i.e. who have a standing bid) at the end of the auction win a unit. The 1^{st} unit is given to the highest bidder, the second unit to the second highest bidder and so on... If the number of standing bids at the end of the auction is less than the number of units auctioned then only the units with a standing bid are sold.

Below are a few examples of how a new bid is treated during a multi-unit GBM auction:

Standing bids (before)

Position	Name	Bid
1	Bidder A	25
2	Bidder B	14
3	Bidder C	13
4	Bidder D	10
5	Bidder E	8

Bidder F bids 10 →

Standing bids (after)

Position	Name	Bid
1	Bidder A	25
2	Bidder B	14
3	Bidder C	13
4	Bidder D	10
5	Bidder F	10

Standing bids (before)

Position	Name	Bid
1	Bidder A	25
2	Bidder B	14
3	Bidder C	13
4	Bidder D	10
5	Bidder E	8

Bidder F bids 27 →

Standing bids (after)

Position	Name	Bid
1	Bidder F	27
2	Bidder A	25
3	Bidder B	14
4	Bidder C	13
5	Bidder D	10

Standing bids (before)

Position	Name	Bid
1	Bidder A	25
2	Bidder B	14
3	Bidder C	13
4	Bidder D	10
5	Bidder E	8

Bidder F bids 12 →

Standing bids (after)

Position	Name	Bid
1	Bidder A	25
2	Bidder B	14
3	Bidder C	13
4	Bidder F	12
5	Bidder D	10

Standing bids (before)

Position	Name	Bid
1	Bidder A	25
2	Bidder B	14
3	Bidder C	13
4	Bidder D	10
5	Bidder E	8

Bidder F bids 11, Bidder G bids 14 →

Standing bids (after)

Position	Name	Bid
1	Bidder A	25
2	Bidder B	14
3	Bidder G	14
4	Bidder C	13
5	Bidder F	11

The revenue of a multi-unit GBM auction is the sum of all k winning bids, minus the incentives paid to bidders that were displaced from the top position. The analysis of the best and worse case revenue are very is similar to the work done in the previous sections for the GBM auction, and the results can be reused.

5.3 Other Incentive Functions

Other incentive functions that the function g defined in section 2 are possible, although these alternatives are inferior.

5.3.1 Fixed incentive amount

The bidders receive the same quantity of money q when outbid, that is $g(x) = q$

Then $\forall i \in [\![1, n-1]\!]$, $g_i b_i = q$ and $R_{\%} = 1 - \frac{(n-1)q}{b_n}$.

This model has many drawbacks, notably the fact that it incentivises bidder to place as small a first bid as possible to maximise the percentage return on that first bid, and then incentivises bidder to bid in increments as small as possible to maximise the number of distributed incentives during the auction.

5.3.2 Fixed incentive percentage

The fiexd percentage is simple the g function defined in section 2, but with the added constraint that $inc_{max} = inc_{min}$ and as a result, $\forall (x, y) \in \mathbb{R}_+^2$, $g(x, y) = inc_{max}$.

We can rewrite the revenue with that added constraint:

$$\begin{aligned} R_{\%} &= 1 - \frac{\sum_{i=1}^{n-1} g_i b_i}{b_n} \\ &= 1 - \frac{\sum_{i=1}^{n-1} inc_{max} b_i}{b_n} \\ &= 1 - inc_{max} \frac{\sum_{i=1}^{n-1} b_i}{b_n} \end{aligned}$$

The pattern for the best case is still the same: the Min $step_{min}$ bidding pattern, except that we can simplify the formula, because $inc_{max} = inc_{min}$:

$$\begin{aligned} R_{\%max} &= 1 - \frac{b_1}{b_n}(inc_{max} + inc_{min}\alpha\frac{\alpha^{n-3} - 1}{\alpha - 1}) \\ &= 1 - \frac{b_1}{b_n}(inc_{max} + inc_{max}\alpha\frac{\alpha^{n-3} - 1}{\alpha - 1}) \\ &= 1 - inc_{max}\frac{b_1}{b_n}(1 + \alpha\frac{\alpha^{n-3} - 1}{\alpha - 1}) \\ &= 1 - inc_{max}\frac{b_1}{b_n}\left(\frac{\alpha + \alpha^{n-2} - \alpha - 1}{\alpha - 1}\right) \\ &= 1 - inc_{max}\frac{b_1}{b_n}\left(\frac{\alpha^{n-2} - 1}{\alpha - 1}\right) \end{aligned}$$

The worst case comes with the Max $step_{min}$ bidding pattern and:

$$\begin{aligned} R_{\%min} &= 1 - \frac{inc_{max} b_1}{b_n} - inc_{min} \frac{\alpha^{n-3} - 1}{\alpha^{n-2}(\alpha - 1)} \\ &= 1 - inc_{max} \left(\frac{b_1}{b_n} + \frac{\alpha^{n-3} - 1}{\alpha^{n-2}(\alpha - 1)} \right) \end{aligned}$$

5.3.3 Non-linear acceleration

In the incentive function g defined in section 2, bidder are not only incentivised to increase their bid because the incentive amount is based on a percentage of their bid, but they are further incentivised because that percentage also growths linearly with the ratio between the two bid (until inc_{max} has been reached). We could envision other incentive function where the growth of the percentage is not linear, but follows an exponential function, for example:

$g(x, y) = inc_{max} \left(1 - e^{-(x - \alpha y)}\right) + inc_{min}$, $0 \leq inc_{max} + inc_{min} \leq 1$
$g(x, y) = inc_{max} \left(1 - \frac{1}{y - \alpha x + \epsilon + 1}\right)$, $\epsilon \in (0, 1)$

The value of such exponential incentive function compared to our chosen linear incentive function g are not clear. With a linear growth, bidders are already clearly incentives to bid closer to their value of the item, and a non-linerar incentive function would prove harder to calculate (and potentially more expensive if calculated on a decentralised network).

5.4 Perpetual GBM Bidding

We envision a mechanism, dubbed Perpetual GBM Bidding, which is an extension of the GBM auction, whereby an item would be perpetually on a GBM auction. Instead of being time-limited, the item is always for sale and receiving offers which bidders can also retract (unlike in the GBM auction), and a sale happens whenever the seller agrees to sell to he current highest bidder. As soon as the sale happens, the item is already back in a perpetual auction state and accepting offers.

The Perpetual GBM Bidding works as follows:

The first bidder to make an offer can put any price he or she wants and will become the highest bidder. The standing offer can only be displaced by a higher offer. Each new offer is required to be higher than the standing offer by a minimum amount. If this new offer is subsequently displaced (by an even higher offer), then the bidder receives an incentive. This incentive is calculated as a percentage of the offer amount, based on the increase that offer represented compared to the previous standing offer. Unlike in a GBM auction, the offer of a specific bidder stands, even after being superseded by a new standing offer, until it is retracted by that bidder, or that bidder makes a higher offer, which will replace his existing offer. Bidders can retract their offer at any time, however, a bidder must pay a cancellation fee if his or her offer has not been superseded by a higher offer, in order to settle incentives distributed since the last settlement. This exception to this rule is the first bidder to make an offer, as he or she can always retract his or her offer for free as no incentives have been distributed. If you have been outbid at least once, then you can retract the full amount of your offer, plus an incentive. As long as there is a standing offer, the seller can at any time accept this standing offer. When accepted, the item being sold or a GBM auction starts with that offer as the standing bid, which can no longer be retracted (but will be able to be displaced by potential higher bids during the auction). The seller can also set an "Auction Trigger" visible to all bidders, which serves as a reserve price. If an offer is made that meets that reserve price, it will automatically be accepted and start a GBM auction with that offer as the standing bid.

This system has several advantages. First, the seller continuously receives offers without the need to mark the item as "For Sale". Bidders are incentivised to make offers, and because offers are fully funded and the bidder will pay a cancellation fee for offers that haven't been superseded, these offers are genuine. Second, the seller also only ever needs to consider the highest offer at any time. If the current highest offer is above a price the seller is willing to sell the item for, by accepting the offer he or she will be guaranteed a minimum proceed. If the current highest offer is not enough for the seller to agree to a sale, he or she can set an auction trigger. Third, if combined with a time-limited GBM auction when the seller accept the highest offer, this system is a great way to prevent black market deals, as the seller does not choose who the winner of the auction is and every bidder has a chance to participate when the seller chooses to sell.

Some of the disadvantages of the Perpetual GBM Bidding include its complexity for the users, the fact that is locks the item in the system in perpetuity (by design), and it prevents instant liquidity if combined with a time-limited GBM auction when the seller accept the highest offer.

5.5 Reserve Price

Reserve prices, as in an amount set by the seller (either publicly or privately) under which the item for sale may not be sold, are very common in auctions nowadays. However, they pose a challenge for the GBM auction, as bidders are paid an incentive when they are outbid, and if the auction does not end in a sale there would be a debt to be paid by the highest bidder, even though that bidder does not win the item. This is clearly not a desirable outcome.

We suggest the following solutions to implement seller protections in a GBM auction without it being seriously detrimental to any stakeholder.

1. Incentives are only paid at the end of the auction if the reserve price has been met. If the sale does not happen (final bid under the reserve price), then no incentives are distributed and everyone leaves as they came in.
2. At the end of the auction, the seller has a "take it or leave it moment", where they can pay the highest bidder their incentive and withdraw the item from sale, or accept the sale.

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